Econ. 511b Spring 1999 George Hall and Chris Sims

Answers for Decentralization and Asset Pricing Exercise*

1. The consumer's first-order conditions deliver

$$\frac{Q_t}{C_1(t)} = \beta E_t \left[\frac{Q_{t+1} + Y_2(t+1)}{C_1(t+1)} \right] . \tag{A1}$$

The consumer's budget constraint is

$$C_1(t) = Y_1(t) + Y_2(t) , \tag{A2}$$

where we have used the fact that $S_1(t) = 1$ for all t. This lets us rewrite (A1) as

$$Q_t = \beta E_t \left[\frac{(Y_1(t) + Y_2(t)) \cdot (Q_{t+1} + Y_2(t+1))}{Y_1(t+1) + Y_2(t+1)} \right] \tag{A3}$$

and then solve it forward to obtain

$$Q_t = E_t \left[(Y_1(t) + Y_2(t)) \sum_{s=1}^{\infty} \beta^s \frac{Y_2(t+s)}{Y_1(t+s) + Y_2(t+s)} \right]$$
$$= (Y_1(t) + Y_2(t)) \frac{E\left[\frac{Y_2(t)}{Y_1(t) + Y_2(t)} \right]}{\beta^{-1} - 1} . \tag{A4}$$

In deriving the last equality in (A4) we are using the fact that the Y's are all i.i.d., so the random ratios in the forward solution all have the same expectation conditional on information at t. So we've now answered the question.

2. The steady state of the model with Y's having mean 1 is $C_1 \equiv C_2 \equiv 1$, with $Q \equiv 1/(\beta^{-1} - 1) \equiv 9$. The four equations we will use are the social resource constraint (49) (in the notes), the individual's constraint (47) with $i = 1$, and the Euler equations with respect to S (54) for $i = 1, 2$, with the Lagrange multiplier substituted out using the C first order conditions (53). The resulting system, arranged to have most recently dated variables on the left (mostly), and with

endogenous error terms inserted to allow us to drop the E_t's, is

$$C_1(t) + C_2(t) = Y_1(t) + Y_2(t) \tag{A5}$$

$$C_1(t) + Q_t S_1(t) - Y_1(t) = (Q_t + \delta_t) S_1(t-1) \tag{A6}$$

$$\beta \frac{Q_t + \delta_t}{C_1(t)} = \frac{Q_{t-1}}{C_1(t-1)} + \eta_1(t) \tag{A7}$$

$$\beta \frac{Q_t + \delta_t}{C_2(t)} = \frac{Q_{t-1}}{C_2(t-1)} + \eta_2(t)\,. \tag{A8}$$

Linearizing these equations around the steady state produces a system in our canonical form

$$\Gamma_0 x_t = \Gamma_1 x_{t-1} + \Psi z_t + \Pi \eta_t \tag{A9}$$

if we define $x_t = [dC_1(t)\; dC_2(t)\; dQ_t\; dS_1(t)]'$ and $z_t = [dY_1(t)\; dY_2(t)\; d\delta_t]'$ and set

$$\Gamma_0 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 9 \\ 9 & 0 & -.9 & 0 \\ 0 & 9 & -.9 & 0 \end{bmatrix}, \qquad \Gamma_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \\ 9 & 0 & -1 & 0 \\ 0 & 9 & -1 & 0 \end{bmatrix},$$

$$\Psi = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & .9 \\ 0 & 0 & .9 \end{bmatrix}, \qquad \Pi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{A10}$$

It is possible, by taking the difference of the last two equations and re-ordering equations and variables, to get this model into a form in which Γ_0 is block triangular, with two 2×2 blocks on the diagonal. It then becomes possible to deal with the system analytically. However, we are going to proceed by brute force numerical methods. The analytic approach would have advantages if we wanted to make a more complete analysis of how the system's properties depend on choices of parameters.

The system matrix has the Jordan decomposition

$$\Gamma_0^{-1}\Gamma_1 = \begin{bmatrix} 0.5000 & -0.5000 & 0 & 0 \\ -0.5000 & 0.5000 & 0 & 0 \\ -5.0000 & -5.0000 & 1.1111 & 0 \\ -0.0556 & 0.0556 & 0.0000 & 1.1111 \end{bmatrix} = V \Lambda V^{-1}\,, \tag{A11}$$

with the eigenvalues arranged along the diagonal of Λ as [1.1111, 1.1111, 1, 0] and

$$V = \begin{bmatrix} 0 & 0 & -0.5774 & 0.1098 \\ 0 & 0 & 0.5774 & 0.1098 \\ 0 & 0.9929 & 0 & 0.9879 \\ 1.0000 & -0.1193 & -0.5774 & -0.0000 \end{bmatrix}. \tag{A12}$$

We will also need to use the matrix whose rows are the left eigenvectors,

$$V^{-1} = \begin{bmatrix} -1.0407 & -0.0407 & 0.1202 & 1.0000 \\ -4.5324 & -4.5324 & 1.0072 & 0 \\ -0.8660 & 0.8660 & 0 & 0 \\ 4.5552 & 4.5552 & 0 & 0 \end{bmatrix} \tag{A13}$$

To find the full solution we have to "solve the unstable equations forward", meaning we make the change of variables $w_t = V^{-1}x_t$, so that the transformed system becomes

$$w_t = \begin{bmatrix} 1.1111 & 0 & 0 & 0 \\ 0 & 1.1111 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} w_{t-1} + V^{-1}\Gamma_0^{-1}\Psi z_t + V^{-1}\Gamma_0^{-1}\Pi\eta_t \,. \tag{A14}$$

Because our linearized system has i.i.d. z's as well as serially uncorrelated η's, the forward solution for the two unstable components of w, w_1 and w_2, has no stochastic component (the expected future values are all identically equal to unconditional means) and simply sets w_1 and w_2 identically equal to their steady state values. In other words,

$$\begin{aligned} \begin{bmatrix} w_{1t} \\ w_{2t} \end{bmatrix} &= \begin{bmatrix} -1.0407 & -0.0407 & 0.1202 & 1.0000 \\ -4.5324 & -4.5324 & 1.0072 & 0 \end{bmatrix} x_t \\ &= V^{1\cdot}\Gamma_0^{-1} E_t \left[\sum_{s=1}^{\infty} \beta^s(\Psi z_{t+s} + \Pi\eta_{t+s}) \right] \\ &= V^{1\cdot}\Gamma_0^{-1} \sum_{s=1}^{\infty} \big(\beta^s(\Psi z_{t+s} + \Pi\eta_{t+s})\big) = 0 \end{aligned} \tag{A15}$$

The full system is then

$$w_t = V^{-1}x_t = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} V^{-1}x_{t-1} + \begin{bmatrix} 0 \\ V^{2\cdot} \end{bmatrix} \Gamma_0^{-1}\Psi z_t + \begin{bmatrix} 0 \\ V^{2\cdot} \end{bmatrix} \Pi\eta_t \,, \tag{A16}$$

where $V^{2\cdot}$ stands for the second pair of rows in V^{-1}. We could solve this system for x by multiplying through by V. However the result would be a system involving η's. We can get rid of the η's by using the fact that the far right-hand side of (A15) must be identically zero, which implies (using the notation that $V^{1\cdot}$ and $V^{2\cdot}$ are the first and second *pairs* of rows in V^{-1}, respectively)

$$V^{1\cdot}\Gamma_0^{-1}(\Psi z_t + \Pi\eta_t) = 0 \tag{A17}$$

$$\begin{bmatrix} 0.1285 & 0.0050 \\ 0.5596 & 0.5596 \end{bmatrix} \eta_t + \begin{bmatrix} 0.1156 & 0.0045 & -0.1202 \\ 0.5036 & 0.5036 & -1.0072 \end{bmatrix} z_t = 0 \tag{A18}$$

$$\eta_t = \begin{bmatrix} -0.9000 & 0.0000 & 0.9000 \\ 0.0000 & -0.9000 & 0.9000 \end{bmatrix} z_t \,. \tag{A19}$$

However, it is a bit more direct to simply combine the first two rows of (A16) with the first two rows of (A9), which do not involve any terms in η. The idea is that, because the solution to the system with the stability conditions imposed is still one solution to the original equations, any set of equations formed from some subset of the equations of the original system plus the equations in (A15) is valid — that is, is satisfied along the solution path. If we can form such an equation system so that it uniquely determines a path for x, it gives us the solution. This approach need not always be available. There may not be enough equations without η's in the original system, or the resulting system may have a singular coefficient matrix on current x_t. But often, as in this example, it does work.

With either approach, the final result, after solving for x_t, is

$$dC_1(t) = .5(dC_1(t-1) - dC_2(t-1)) + .55dY_1(t) + .45dY_2(t) \tag{A20}$$

$$dC_2(t) = -.5(dC_1(t-1) - dC_2(t-1) + .45dY_1(t) + .55dY_2(t) \tag{A21}$$

$$dQ_t = 4.5(dY_1(t) + dY_2(t)) \tag{A22}$$

$$dS_t = .5(dC_1(t) - dC_2(t)) + .05(dY_1(t) - dY_2(t)) \,. \tag{A23}$$

This differs from the complete-markets linearization, given in the problem statement, in that

1. It makes C_1 respond somewhat more strongly than C_2 to Y_1 shocks, and vice versa for Y_2 shocks, while the complete markets solution makes them respond equally.
2. It introduces serial correlation, so that when C_1 was higher than C_2 last period, it tends to remain so this period. The logic of this is that in this incomplete markets model the agents' relative wealths are subject to random variation. An agent consumes relatively more when the agent is relatively more wealthy, and wealth differences tend to be serially correlated.

Note that fluctuations in δ_t have no effect at all in this linearization, regardless of what stochastic process they follow. This is a special property of the linearization in the neighborhood of $S = 0$. If we linearized around a steady state with $S \neq 0$, we would find that shocks to δ affected relative wealths and thereby consumption paths.

Note also that the linearization implies $dS \equiv .5(dC_1 - dC_2)$. So (A20)-(A23) above could be written with dS replacing all the $.5(dC_1 - dC_2)$ terms.